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Additive decompositions of θ-functions of multiple arguments

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Abstract. Product decompositions of θ-functions of multiple argument are well known in the literature of the subject. Here, additive decompositions are presented.

θ-functions were first introduced by Jacobi as a means of calculating elliptic functions. They are functions of a complex variable, z, and a parameter $q = e^{i\pi\tau}$. They are denoted here $\theta(z|\tau)$. Four types of θ-function were considered by Jacobi, and both infinite product and infinite series representations of each θ-function are known (Whittaker and Watson 1958), e.g.

$$\theta_4(z|\tau) = Q_0\prod_1^\infty(1-2q^{2r-1}\cos 2z + q^{4r-2}) = \sum_{-\infty}^{\infty}(-1)^r q^{r^2}\cos 2rz \tag{1}$$

where $Q_0 = \prod_1^\infty (1-q^{2r})$.

The decomposition of θ-functions of multiple argument into *products* of θ-functions of simpler argument is well (if not widely) known. They may be found in standard treatises, e.g. Tannery and Molk (1972). For example

$$\theta_4(nz|n\tau) = Q_0(q^n)\prod_1^\infty(1-2q^{(2r-1)n}\cos 2nz + q^{(4r-2)n}). \tag{2}$$

Now by a theorem of Cotes

$$(1-2q^{(2r-1)n}\cos 2nz + q^{(4r-2)n}) = \prod_{s=0}^{n-1}(1-2q^{2r-1}\cos(2z+2s\pi/n)+q^{4r-2}]. \tag{3}$$

Hence

$$\theta_4(nz|n\tau) = \frac{Q_0(q^n)}{Q_0^n}\prod_{s=0}^{n-1}\theta_4\left(z+\frac{s\pi}{n}\middle|\tau\right). \tag{4}$$

Similar results may be obtained for θ_1, θ_2 and θ_3, and all may be summarised in the one equation

$$\theta_\mu(nz|n\tau) = \frac{Q_0(q^n)}{Q_0^n}\prod_{s=\alpha}^{s=\beta}\theta_\mu\left(z+\frac{s\pi}{n}\middle|\tau\right), \tag{5}$$

$$\left.\begin{array}{l}\alpha = 0\\ \beta = n-1\end{array}\right\}\mu = 1, 4, \qquad \left.\begin{array}{l}\alpha = -(n-1)/2\\ \beta = (n-1)/2\end{array}\right\}\mu = 2,3.$$

These results may be regarded as the elliptic function analogues of the Cotes identities for the circular functions, e.g.

$$\sin nz = 2^{n-1} \prod_{s=0}^{n-1} \sin (z + s\pi/n). \qquad (6)$$

In a recent investigation of rational von Neumann lattices (Boon *et al* 1982), the decription of harmonic oscillator states on such lattices using the kq-representation (Zak 1968) led to an *additive* decomposition of $\theta_3(nz|n\tau)$. This was

$$n\theta_3(nz|n\tau) = \sum_{s=0}^{n-1} \theta_3\left(z + \frac{s\pi}{n}\middle|\frac{\tau}{n}\right). \qquad (7)$$

Corresponding results for the other θ-functions are

$$n\theta_4(nz|n\tau) = \sum_{s=0}^{n-1} \theta_3\left(z + (2s+1)\frac{\pi}{n}\middle|\frac{\tau}{n}\right),$$

$$\left.\begin{aligned} n\theta_1(nz|n\tau) &= \sum_{s=0}^{n-1} (-1)^s\theta_3\left(z + (2s+1)\frac{\pi}{n}\middle|\frac{\tau}{n}\right) \\ n\theta_2(nz|n\tau) &= \sum_{s=0}^{n-1} (-1)^s\theta_3\left(z + \frac{s\pi}{n}\middle|\frac{\tau}{n}\right) \end{aligned}\right\} \quad n \text{ even}. \qquad (8)$$

We have not been able to find these results in any standard text on θ-functions, and thus present them here.

Simple direct proofs of these results may be obtained from the definitions of θ-functions as infinite series. Thus (7) may be derived as follows:

$$\theta_3(z|\tau) = \sum_{-\infty}^{\infty} q^{r^2} \cos 2rz = \sum_{-\infty}^{\infty} \exp(\mathrm{i}\pi r^2\tau) \exp(2\mathrm{i}rz).$$

Hence

$$\sum_{s=0}^{n-1} \theta_3\left(z + \frac{s\pi}{n}\middle|\frac{\tau}{n}\right) = \sum_{-\infty}^{\infty} \exp\left(\mathrm{i}\pi r^2\frac{\tau}{n} + 2\mathrm{i}rz\right) \sum_{s=0}^{n-1} \exp\left(\frac{2\mathrm{i}rs\pi}{n}\right)$$

$$= \sum_{-\infty}^{\infty} \exp\left(\frac{\mathrm{i}\pi r^2\tau}{n} + 2\mathrm{i}rz\right)\left(n \sum_{m=-\infty}^{\infty} \delta_{r,mn}\right)$$

$$= n \sum_{m=-\infty}^{\infty} \exp(\mathrm{i}\pi m^2 n\tau + 2\mathrm{i}mnz) = n\theta_3(nz|n\tau). \qquad (9)$$

It seems possible that similar additive decompositions might be found for powers of θ-functions by using various Jacobi identities. Thus one has

$$2n\theta_3(0/\tau)\theta_3(nz|n^2\tau) = \sum_{s=0}^{2n-1} \theta_3^2\left(\frac{z}{2} + \frac{s\pi}{2n}\middle|\frac{\tau}{2}\right). \qquad (10)$$

No doubt many other such identities may be formulated.

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